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ECE 204 *Numerical methods*

Intermediate-value theorem

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Introduction

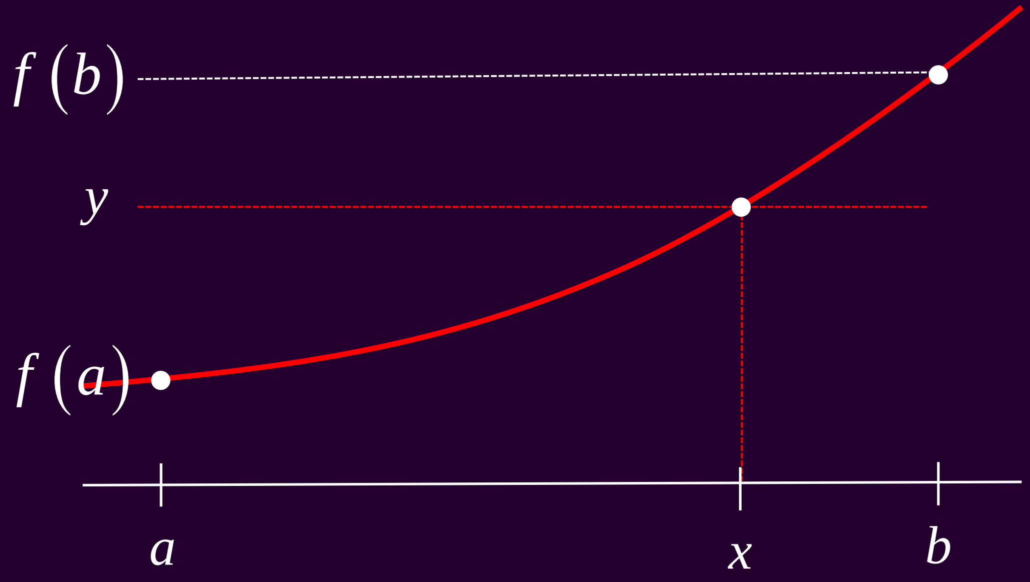
- In this topic, we will
  - Review the intermediate-value theorem
  - Look at some theorems that result from this theorem
  - Look at applications of these new theorems
  - Discuss how we will apply these theorems in this course

The intermediate-value theorem

- You may recall the intermediate-value theorem from calculus

Theorem

Given a continuous function f defined on an interval $[a, b]$,
if y is any value between $f(a)$ and $f(b)$,
then there exists an x such that $a < x < b$ and $y = f(x)$



Application

Theorem

Given a continuous function f defined on an interval $[a, b]$

where $f(a)$ and $f(b)$ have opposite signs,

then there exists a root x such that $a < x < b$

Proof:

If $f(a)$ and $f(b)$ have opposite signs, then 0 is a value between $f(a)$ and $f(b)$.

Therefore, by the intermediate-value theorem, there exists a point x such that $a < x < b$ and $f(x) = 0$.

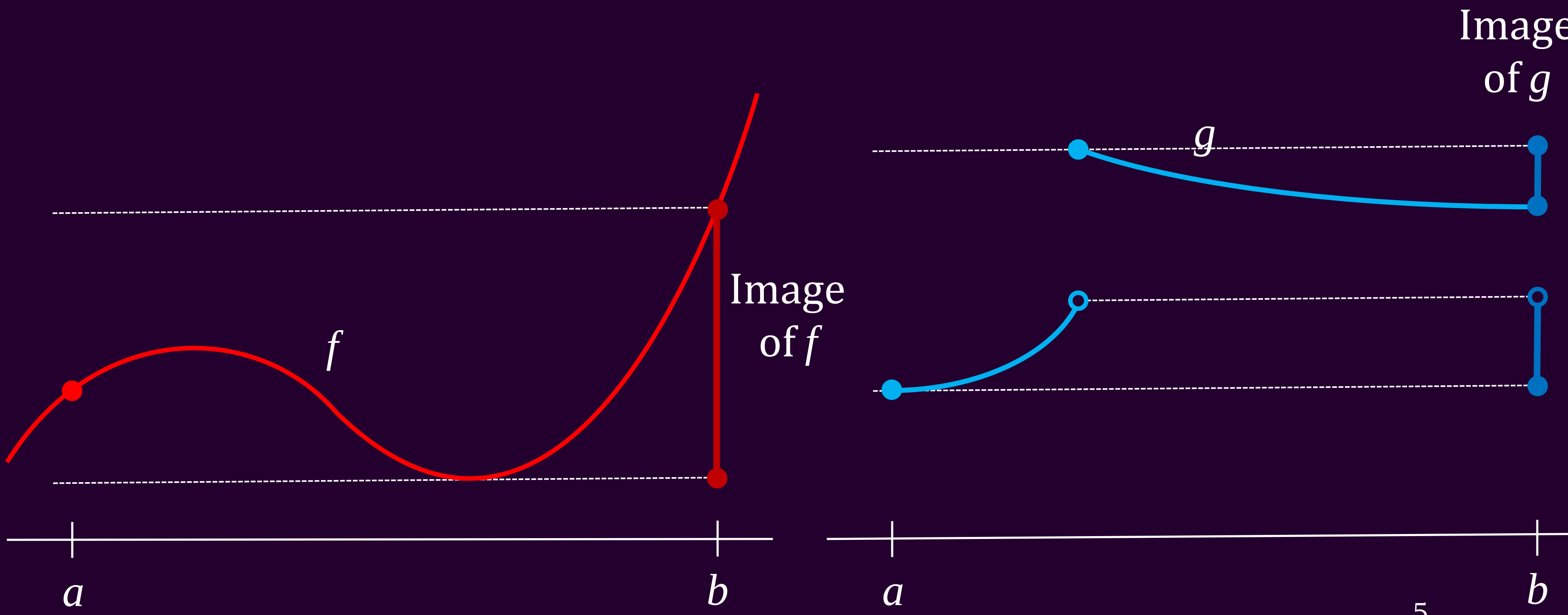
Therefore, f has a root on the interval (a, b). █

Application

- Definition

  Given a function f defined on an interval $[a, b]$,

  the image of $[a, b]$ under f is the collection of all y-values such that there exists an $a \leq x \leq b$ where $y = f(x)$.



Application

Theorem

Given a continuous function f defined on an interval $[a, b]$ the image of $[a, b]$ under f is also an interval.

Proof:

Suppose that the image of $[a, b]$ is not an interval.

Therefore, the image must be split by at least one value y

such that there exists one $y_L < y$ and one $y < y_U$ in the image.

Thus, there exist points x_L and x_U in $[a, b]$ such that

$$f(x_L) = y_L \text{ and } f(x_U) = y_U$$

But either $[x_L, x_U]$ or $[x_U, x_L]$ is a sub-interval of $[a, b]$, and by the intermediate-value theorem, there exists an x between these two such that $f(x) = y$.

But this x must also be in the interval $[a, b]$, and so y is in the image.

This is a contradiction.

Therefore the continuous image of an interval is an interval. █

Application

Theorem

Suppose f is a continuous function on $[a, b]$ and we have two x-values, x_1 and x_2, also on the interval $[a, b]$.

Then there exists an x in $[a, b]$ such that

$$f(x) = \frac{f(x_1) + f(x_2)}{2}$$

Proof:

$f(x_1)$ and $f(x_2)$ are both in the image of $[a, b]$ under f.

The average is between these two.

From the previous theorem, therefore, there must be an x that satisfies the given condition. █

Application

Theorem

Suppose f is a continuous function on $[a, b]$ and we have n x-values x_1 through x_n all from the interval $[a, b]$.

Then there must exist an x such that $f(x)$ equals a given convex combination, that is, a weighted average with all weights being non-negative: $w_1 f(x_1) + \cdots + w_n f(x_n)$

Proof:

We saw previously that any convex combination must satisfy

$$\min\{f(x_1), \ldots, f(x_n)\} \le w_1 f(x_1) + \cdots + w_n f(x_n) \le \max\{f(x_1), \ldots, f(x_n)\}$$

As both end-points are in the image, thus there must be an x in $[a, b]$ such that

$$f(x) = w_1 f(x_1) + \cdots + w_n f(x_n) \cdot \blacksquare$$

Example

- Suppose we have the interval [0, 1] and we have the nine points

  0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9
- Suppose also that the weights of our weighted average are

  0.05, 0.01, 0.42, 0.03, 0.14, 0.09, 0.18, 0.02, 0.06

  – We can calculate that

$$0.05\sin(0.1)+0.01\sin(0.2)+0.42\sin(0.3)+0.03\sin(0.4)+0.14\sin(0.5)$$
$$+0.09\sin(0.6)+0.18\sin(0.7)+0.02\sin(0.8)+0.06\sin(0.9)$$
$$\approx 0.4380227193158797$$

  – Note that $\sin(0.4533979810675964) \approx 0.4380227193158797$ and $0 < 0.4533979810675964 < 1$

In this course

- In general, we will only use these theorems to determine that a particular value exists
  - We often will not go out looking for the actual value

- For example,

$$f(x+h) = f(x) + f^{(1)}(x)h + \frac{1}{2}f^{(2)}(\xi_+)h^2$$

$$f(x-h) = f(x) - f^{(1)}(x)h + \frac{1}{2}f^{(2)}(\xi_-)h^2$$

$$\frac{f(x+h)+f(x-h)}{2} = f(x) + \boxed{\frac{1}{4}\left(f^{(2)}(\xi_+)+f^{(2)}(\xi_-)\right)}h^2$$

$$\frac{1}{2}\left(\frac{1}{2}f^{(2)}(\xi_+)+\frac{1}{2}f^{(2)}(\xi_-)\right) = \frac{1}{2}f^{(2)}(\xi)$$

$$x-h \le \xi \le x+h$$

Example

- To show this result in action:

$$\frac{f(x+h)+f(x-h)}{2} = f(x)+\frac{1}{2}f^{(2)}(\xi)h^2$$

  - Let the function be the cosine function and let $x = 1$ and $h = 0.5$
  - Then

$$\frac{\cos(0.5)+\cos(1.5)}{2} = \cos(1)-\frac{1}{2}\cos(1.30303863520662\,2)0.5^2$$

$$0.5 \le 1.01321030987291\,4 \le 1.5$$

Summary

- Following this topic, you now
  - Have reviewed the intermediate-value theorem
  - Seen a few theorems that are a consequence of that theorem
  - Seen a few applications these theorems and how these will be used in this course

References

[1] https://en.wikipedia.org/wiki/Intermediate_value_theorem

Acknowledgments

Mayuresh Gaikwad for noting an error on p.9, which propagated to p.10.

Tazik Shahjahan for pointing out typos.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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